

Continuous orbit equivalence of topological Markov shifts and dynamical zeta functions

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Abstract

For continuously orbit equivalent one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) , their eventually periodic points and cocycle functions are studied. As a result we directly construct an isomorphism between their ordered cohomology groups (\bar{H}^A, \bar{H}_+^A) and (\bar{H}^B, \bar{H}_+^B) . We also show that the cocycle functions for the continuous orbit equivalences give rise to positive elements of the ordered cohomology, so that the zeta functions of continuously orbit equivalent topological Markov shifts are related. The set of Borel measures is shown to be invariant under continuous orbit equivalence of one-sided topological Markov shifts.

1 Introduction

Let A be an irreducible square matrix with entries in $\{0, 1\}$. Denote by \bar{X}_A the shift space of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ for A . The ordered cohomology group (\bar{H}^A, \bar{H}_+^A) is defined by the quotient group of the ordered abelian group $C(\bar{X}_A, \mathbb{Z})$ of all \mathbb{Z} -valued continuous functions on \bar{X}_A quoted by the subgroup $\{\xi - \xi \circ \bar{\sigma}_A \mid \xi \in C(\bar{X}_A, \mathbb{Z})\}$. The positive cone \bar{H}_+^A consists of the classes of nonnegative functions in $C(\bar{X}_A, \mathbb{Z})$ (cf. [1], [13]). We similarly define the ordered cohomology group (H^A, H_+^A) for one-sided topological Markov shift (X_A, σ_A) . The latter ordered group (H^A, H_+^A) is naturally isomorphic to the former one (\bar{H}^A, \bar{H}_+^A) ([8, Lemma 3.1]). In [1], Boyle-Handelman have proved that the ordered cohomology group (\bar{H}^A, \bar{H}_+^A) is a complete invariant for flow equivalence of two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. Continuous orbit equivalence of one-sided topological Markov shifts is regarded as a counterpart for flow equivalence of two-sided topological Markov shifts (see [8, Theorem 2.3], [8, Corollary 3.8]). It is closely related to the classifications of both the étale groupoids associated to the one-sided Markov shifts and the Cuntz-Krieger algebras (see [6], [7], [8], [9], [10], cf. [3], [4], [14], [17]). By using the above Boyle-Handelman's result, it has been proved in [8] that continuous orbit equivalence of one-sided topological Markov shift (X_A, σ_A) yields flow equivalence of their two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$. By Parry-Sullivan [12], this implies that the determinant $\det(\text{id} - A)$ is invariant under continuous orbit equivalence of one-sided topological Markov shift (X_A, σ_A) . Let G_A denote the étale groupoid for (X_A, σ_A) whose reduced groupoid C^* -algebra $C_r^*(G_A)$ is isomorphic to the Cuntz-Krieger algebra \mathcal{O}_A (see [8], [9], [10]). As a result, it has been shown that the following three assertions for one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are equivalent ([8, Theorem 3.6]):

- (i) (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.
- (ii) The étale groupoids G_A and G_B are isomorphic.
- (iii) The Cuntz-Krieger algebras \mathcal{O}_A and \mathcal{O}_B are isomorphic and $\det(\text{id} - A) = \det(\text{id} - B)$.

The method in [8] by which continuous orbit equivalence of one-sided topological Markov shifts yields flow equivalence of the two-sided topological Markov shifts has been due to a technique of the groupoids associated with the one-sided topological Markov shifts.

In this paper, we will study eventually periodic points and cocycle functions of continuously orbit equivalent one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) . We then directly construct an isomorphism between their ordered cohomology groups (H^A, H_+^A) and (H^B, H_+^B) without using groupoid. Let A, B be square irreducible matrices with entries in $\{0, 1\}$. Suppose that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \rightarrow X_B$ so that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A, \quad (1.1)$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B \quad (1.2)$$

for some continuous functions $k_1, l_1 \in C(X_A, \mathbb{Z})$, $k_2, l_2 \in C(X_B, \mathbb{Z})$. We will directly construct a map $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ which yields an isomorphism from H^B to H^A as abelian groups. We call the functions $c_1(x) = l_1(x) - k_1(x)$, $x \in X_A$ and $c_2(y) = l_2(y) - k_2(y)$, $y \in X_B$ the cocycle functions for h and h^{-1} respectively. We will prove that the classes $[c_1]$ in H^A and $[c_2]$ in H^B of c_1 and c_2 give rise to positive elements in the ordered groups (H^A, H_+^A) and (H^B, H_+^B) respectively. By using the positivities of $[c_1]$ and $[c_2]$, we will show that $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ induces an isomorphism of the ordered groups from (H^B, H_+^B) to (H^A, H_+^A) (Theorem 5.11, cf. [8]).

Continuous orbit equivalence relation of one-sided topological Markov shifts preserve their eventually periodic points (Proposition 3.5), so that the sets of periodic orbits of the associated two-sided topological Markov shifts are preserved. Hence there are some relation between their zeta functions. We will show that the dynamical zeta function $\zeta_{[c_1]}(t)$ for the cocycle function c_1 coincides with the zeta function $\zeta_B(t)$ of the Markov shift $(\bar{X}_B, \bar{\sigma}_B)$ (Theorem 6.7). Namely

$$\zeta_{[c_1]}(t) = \zeta_B(t) \quad \text{and similarly} \quad \zeta_{[c_2]}(t) = \zeta_A(t).$$

It is well-known that periodic points of a transformation gives rise to invariant probability measures on the space by averaging the point mass. As the continuous orbit equivalence preserves structure of periodic orbits, it is reasonable to have a relationship between their shift-invariant measures. We will show that there exists an order isomorphism between the set of σ_A -invariant regular Borel measures on X_A and the set of σ_B -invariant regular Borel measures on X_B . If in particular, the class $[c_1]$ (resp. $[c_2]$) of the cocycle function c_1 (resp. c_2) is cohomologous to 1 in H^A (resp. H^B), there exists an affine isomorphism between the set of σ_A -invariant regular Borel probability measures on X_A and the set of σ_B -invariant regular Borel probability measures on X_B (Theorem 7.2). Hence the set of shift-invariant regular Borel measures on the one-sided topological Markov shift is invariant under continuous orbit equivalence.

Throughout the paper, we denote by \mathbb{N} the set of positive integers and by \mathbb{Z}_+ the set of nonnegative integers, respectively.

2 Preliminaries

Let $A = [A(i, j)]_{i, j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we assume that A has no rows or columns identically equal to zero. We denote by X_A the shift space

$$X_A = \{(x_n)_{n \in \mathbb{N}} \in \{1, \dots, N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

of the right one-sided topological Markov shift for A . It is a compact Hausdorff space in natural product topology on $\{1, \dots, N\}^{\mathbb{N}}$. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$ is a continuous surjective map on X_A . The topological dynamical system (X_A, σ_A) is called the (right) one-sided topological Markov shift for A . We henceforth assume that A is irreducible and satisfies condition (I) in the sense of Cuntz–Krieger [2].

A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1, \dots, N\}$ is said to be admissible for X_A if μ appears in somewhere in an element x in X_A . The length of μ is k and denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length k . We set $B_*(X_A) = \bigcup_{k=0}^{\infty} B_k(X_A)$ where $B_0(X_A)$ denotes the empty word \emptyset . For $x = (x_n)_{n \in \mathbb{N}} \in X_A$ and $k, l \in \mathbb{N}$ with $k \leq l$, we set

$$\begin{aligned} x_{[k, l]} &= x_k x_{k+1} \cdots x_l \in B_{l-k+1}(X_A), \\ x_{[k, \infty)} &= (x_k, x_{k+1}, \dots) \in X_A. \end{aligned}$$

For $x = (x_n)_{n \in \mathbb{N}} \in X_A$, the orbit $\text{orb}_{\sigma_A}(x)$ of x under σ_A is defined by

$$\text{orb}_{\sigma_A}(x) = \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \subset X_A.$$

Let (X_A, σ_A) and (X_B, σ_B) be two topological Markov shifts. If there exists a homeomorphism $h : X_A \rightarrow X_B$ such that $h(\text{orb}_{\sigma_A}(x)) = \text{orb}_{\sigma_B}(h(x))$ for $x \in X_A$, then (X_A, σ_A) and (X_B, σ_B) are said to be topologically orbit equivalent. In this case, one has $h(\sigma_A(x)) \in \bigcup_{k=0}^{\infty} \bigcup_{l=0}^{\infty} \sigma_B^{-k}(\sigma_B^l(h(x)))$ for $x \in X_A$. Hence there exist $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \quad (2.1)$$

Similarly there exist $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ such that

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_B. \quad (2.2)$$

If we may take $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ as continuous maps, the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *continuously orbit equivalent*. If two one-sided topological Markov shifts are topologically conjugate, one may take $k_1(x) = k_2(y) = 0$ and $l_1(x) = l_2(y) = 1$ so that they are continuously orbit equivalent.

For the two matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent, but not topologically conjugate (see [6, Section 5]).

Throughout the paper, we assume that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. We fix a homeomorphism $h : X_A \rightarrow X_B$ and continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying the equalities (2.1) and (2.2).

3 Eventually periodic points

In this section, we will show that the set of eventually periodic points is invariant under continuous orbit equivalence. For $n \in \mathbb{N}$, put

$$\begin{aligned} k_1^n(x) &= \sum_{i=0}^{n-1} k_1(\sigma_A^i(x)), & l_1^n(x) &= \sum_{i=0}^{n-1} l_1(\sigma_A^i(x)), & x &\in X_A, \\ k_2^n(y) &= \sum_{i=0}^{n-1} k_2(\sigma_B^i(y)), & l_2^n(y) &= \sum_{i=0}^{n-1} l_2(\sigma_B^i(y)), & y &\in X_B. \end{aligned}$$

We note that the following identities hold.

Lemma 3.1.

$$k_1^{n+m}(x) = k_1^n(x) + k_1^m(\sigma_A^n(x)), \quad x \in X_A, \quad (3.1)$$

$$l_1^{n+m}(x) = l_1^n(x) + l_1^m(\sigma_A^n(x)), \quad x \in X_A, \quad (3.2)$$

$$k_2^{n+m}(y) = k_2^n(y) + k_2^m(\sigma_B^n(y)), \quad y \in X_B, \quad (3.3)$$

$$l_2^{n+m}(y) = l_2^n(y) + l_2^m(\sigma_B^n(y)), \quad y \in X_B, \quad (3.4)$$

and

$$\sigma_B^{k_1^n(x)}(h(\sigma_A^n(x))) = \sigma_B^{l_1^n(x)}(h(x)), \quad x \in X_A, \quad (3.5)$$

$$\sigma_A^{k_2^n(y)}(h^{-1}(\sigma_B^n(y))) = \sigma_A^{l_2^n(y)}(h^{-1}(y)), \quad y \in X_B. \quad (3.6)$$

Lemma 3.2. *Keep the above notations.*

(i) *If $x, z \in X_A$ satisfy $\sigma_A^p(x) = \sigma_A^q(z)$ for some $p, q \in \mathbb{Z}_+$, then we have*

$$\sigma_B^{l_1^p(x)+k_1^q(z)}(h(x)) = \sigma_B^{k_1^p(x)+l_1^q(z)}(h(z)).$$

(ii) *If $y, w \in X_B$ satisfy $\sigma_B^r(y) = \sigma_B^s(w)$ for some $r, s \in \mathbb{Z}_+$, then we have*

$$\sigma_A^{l_2^r(y)+k_2^s(w)}(h^{-1}(y)) = \sigma_A^{k_2^r(y)+l_2^s(w)}(h^{-1}(w)).$$

Proof. (i) Put $u = \sigma_A^p(x) = \sigma_A^q(z) \in X_A$. It follows that by (3.5)

$$\sigma_B^{l_1^p(x)}(h(x)) = \sigma_B^{k_1^p(x)}(h(\sigma_A^p(x))) = \sigma_B^{k_1^p(x)}(h(u)),$$

and similarly $\sigma_B^{l_1^q(z)}(h(z)) = \sigma_B^{k_1^q(z)}(h(u))$ so that

$$\sigma_B^{l_1^p(x)+k_1^q(z)}(h(x)) = \sigma_B^{k_1^q(z)+k_1^p(x)}(h(u)) = \sigma_B^{k_1^p(x)+l_1^q(z)}(h(z)).$$

(ii) is similarly shown to (i). □

A point $x \in X_A$ is said to be eventually periodic if there exist $p, q \in \mathbb{Z}_+$ with $p \neq q$ such that $\sigma_A^p(x) = \sigma_A^q(x)$. The number $|p - q|$ is called the eventual period of x . The least number of the eventual periods of x is called the least eventual period of x . If in particular $\sigma_A^p(x) = x$ for some $p \in \mathbb{N}$, $x \in X_A$ is said to be p -periodic.

The two identities (i) and (ii) in the following lemma play important rôle in our further discussions.

Lemma 3.3. For $x \in X_A$, $y \in X_B$ and $p \in \mathbb{Z}_+$, we have

$$(i) \quad k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p = k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x)).$$

$$(ii) \quad k_1^{l_2^p(y)}(h^{-1}(y)) + l_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + p = k_1^{k_2^p(y)}(h^{-1}(\sigma_B^p(y))) + l_1^{l_2^p(y)}(h^{-1}(y)).$$

Proof. (i) Put $n = l_1^p(x)$, $m = k_1^p(x)$. By (3.5), one has

$$h^{-1}(\sigma_B^m(h(\sigma_A^p(x)))) = h^{-1}(\sigma_B^n(h(x))). \quad (3.7)$$

By applying $\sigma_A^{k_2^m(h(\sigma_A^p(x))) + k_2^n(h(x))}$ to (3.7), one has

$$\sigma_A^{k_2^m(h(\sigma_A^p(x))) + k_2^n(h(x))}(h^{-1}(\sigma_B^m(h(\sigma_A^p(x))))) = \sigma_A^{k_2^m(h(\sigma_A^p(x))) + k_2^n(h(x))}(h^{-1}(\sigma_B^n(h(x)))). \quad (3.8)$$

The left hand side of (3.8) goes to

$$\sigma_A^{k_2^n(h(x))}(\sigma_A^{k_2^m(h(\sigma_A^p(x)))}(h^{-1}(\sigma_B^m(h(\sigma_A^p(x))))) = \sigma_A^{k_2^n(h(x)) + l_2^m(h(\sigma_A^p(x))) + p}(x).$$

The right hand side of (3.8) goes to

$$\sigma_A^{k_2^m(h(\sigma_A^p(x)))}(\sigma_A^{k_2^n(h(x))}(h^{-1}(\sigma_B^n(h(x))))) = \sigma_A^{k_2^m(h(\sigma_A^p(x))) + l_2^n(h(x))}(x).$$

Hence we have

$$\sigma_A^{k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p}(x) = \sigma_A^{k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x))}(x).$$

Now suppose that there exists $x \in X_A$ such that

$$k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p \neq k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x)) \quad (3.9)$$

so that x is an eventually periodic point. Since the functions k_1, l_1, k_2, l_2 are all continuous, (3.9) hold for all elements of a neighborhood of x . Hence there exists an open set of X_A whose elements are all eventually periodic points. It is a contradiction to the fact that the set of all non eventually periodic points is dense in X_A . Therefore the identity

$$k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(\sigma_A^p(x))) + p = k_2^{k_1^p(x)}(h(\sigma_A^p(x))) + l_2^{l_1^p(x)}(h(x))$$

holds for all $x \in X_A$.

(ii) is similarly shown to (i). □

Lemma 3.4. Let x be a periodic point in X_A . Then $h(x)$ is an eventually periodic point in X_B .

Proof. Assume that $\sigma^p(x) = x$ for some $p \in \mathbb{N}$. By the above lemma (i), we have

$$k_2^{l_1^p(x)}(h(x)) + l_2^{k_1^p(x)}(h(x)) + p = k_2^{k_1^p(x)}(h(x)) + l_2^{l_1^p(x)}(h(x))$$

so that $l_1^p(x) \neq k_1^p(x)$. By the identity (3.5) with $\sigma_A^p(x) = x$, one has

$$\sigma_B^{k_1^p(x)}(h(x)) = \sigma_B^{l_1^p(x)}(h(x)) \quad (3.10)$$

which implies that $h(x)$ is an eventually periodic point in X_B . □

Proposition 3.5. *Suppose that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent via a homeomorphism $h : X_A \rightarrow X_B$ satisfying the equalities (2.1) and (2.2). Let x be an eventually periodic point in X_A . Then $h(x)$ is an eventually periodic point in X_B . Therefore the set of eventually periodic points of a one-sided topological Markov shift is invariant under continuous orbit equivalence.*

Proof. Let x be an eventually periodic point in X_A such that $\sigma_A^{p+q}(x) = \sigma_A^p(x)$ for some $p \in \mathbb{Z}_+, q \in \mathbb{N}$. Put $\tilde{x} = \sigma_A^p(x)$. By the preceding lemma, $h(\tilde{x})$ is an eventually periodic point in X_B . Take $p_1, p_2 \in \mathbb{Z}_+$ with $p_1 \neq p_2$ such that $\sigma_B^{p_1}(h(\tilde{x})) = \sigma_B^{p_2}(h(\tilde{x}))$. By Lemma 3.2, there exist $q_1, q_2 \in \mathbb{Z}_+$ such that $\sigma_B^{q_1}(h(x)) = \sigma_B^{q_2}(h(\tilde{x}))$, so that we have

$$\sigma_B^{p_1+q_1}(h(x)) = \sigma_B^{p_1}(\sigma_B^{q_2}(h(\tilde{x}))) = \sigma_B^{p_2}(\sigma_B^{q_2}(h(\tilde{x}))) = \sigma_B^{p_2+q_1}(h(x)).$$

Since $p_1 + q_1 \neq p_2 + q_1$, $h(x)$ is an eventually periodic point in X_B . \square

4 Construction of an isomorphism from $C(X_B, \mathbb{Z})$ to $C(X_A, \mathbb{Z})$

We are assuming that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent through a homeomorphism h from X_A to X_B with continuous functions $k_1, l_1 : X_A \rightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \rightarrow \mathbb{Z}_+$ satisfying (2.1) and (2.2). Let us denote by $C(X_A, \mathbb{Z})$ (resp. $C(X_B, \mathbb{Z})$) the abelian group of all integer valued continuous functions on X_A (resp. X_B). In this section, we will directly construct an isomorphism from $C(X_B, \mathbb{Z})$ to $C(X_A, \mathbb{Z})$ compatible to the shifts.

Lemma 4.1. *The function $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ does not depend on the choice of $k_1, l_1 \in C(X_A, \mathbb{Z})$ as long as satisfying (2.1).*

Proof. Let $k'_1, l'_1 \in C(X_A, \mathbb{Z})$ be another functions for k_1, l_1 satisfying

$$\sigma_B^{k'_1(x)}(h(\sigma_A(x))) = \sigma_B^{l'_1(x)}(h(x)) \quad \text{for } x \in X_A. \quad (4.1)$$

Since k_1, k'_1 are both continuous, there exists $K \in \mathbb{N}$ such that $k_1(x), k'_1(x) \leq K$ for all $x \in X_A$. Put $c'_1(x) = l'_1(x) - k'_1(x)$ for $x \in X_A$ so that

$$\sigma_B^{c_1(x)+K}(h(x)) = \sigma_B^{c'_1(x)+K}(h(x)) \quad \text{for all } x \in X_A.$$

Suppose that $c_1(x_0) \neq c'_1(x_0)$ for some $x_0 \in X_A$. There exists a clopen neighborhood U of x_0 such that $c_1(x) \neq c'_1(x)$ for all $x \in U$. As $c_1(x) + K \neq c'_1(x) + K$ for all $x \in U$, the points $h(x)$ for all $x \in U$ are eventually periodic points, which is a contradiction to the fact that the set of non eventually periodic points is dense in X_B . Therefore we conclude that $c_1(x) = c'_1(x)$ for all $x \in X_A$. \square

For $f \in C(X_B, \mathbb{Z})$, define

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) \quad x \in X_A. \quad (4.2)$$

It is easy to see that $\Psi_h(f) \in C(X_A, \mathbb{Z})$. Thus $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ gives rise to a homomorphism of abelian groups.

Lemma 4.2. $\Psi_h : C(X_B, \mathbb{Z}) \longrightarrow C(X_A, \mathbb{Z})$ does not depend on the choice of the functions k_1, l_1 as long as satisfying (2.1).

Proof. Let $k'_1, l'_1 \in C(X_A, \mathbb{Z})$ be another functions for k_1, l_1 satisfying (4.1). We fix an arbitrary $x \in X_A$. By Lemma 4.1, we see that $l_1(x) - k_1(x) = l'_1(x) - k'_1(x)$. Assume that $l'_1(x) < l_1(x)$ so that $k_1(x) - k'_1(x) = l_1(x) - l'_1(x) > 0$. We put

$$\Psi'_h(f)(x) = \sum_{i=0}^{l'_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k'_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) ,$$

so that

$$\Psi_h(f)(x) - \Psi'_h(f)(x) = \sum_{i=l'_1(x)}^{l_1(x)-1} f(\sigma_B^i(h(x))) - \sum_{j=k'_1(x)}^{k_1(x)-1} f(\sigma_B^j(h(\sigma_A(x)))) .$$

As $\sigma_B^{l'_1(x)+j}(h(x)) = \sigma_B^{k'_1(x)+j}(h(\sigma_A(x)))$ for $j = 0, 1, \dots, l_1(x) - l'_1(x) - 1 (= k_1(x) - k'_1(x) - 1)$, we see that $\Psi_h(f)(x) - \Psi'_h(f)(x) = 0$. \square

The equalities in the following lemma are basic in our further discussions.

Lemma 4.3. For $f \in C(X_B, \mathbb{Z})$, $x \in X_A$ and $m = 1, 2, \dots$, the following equalities hold:

$$\begin{aligned} & \sum_{i=0}^{m-1} \left\{ \sum_{i'=0}^{l_1(\sigma_A^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{i+1}(x)))) \right\} \\ &= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))) . \end{aligned}$$

Proof. For $m = 1$, the left hand side of the desired equality is

$$\sum_{i'=0}^{l_1(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1(x)-1} f(\sigma_B^{j'}(h(\sigma_A(x)))) .$$

which is equal to the right hand side of the desired equality.

We assume that the desired formula holds for some m . It then follows that

$$\begin{aligned}
& \sum_{i=0}^m \left\{ \sum_{i'=0}^{l_1(\sigma_A^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{i+1}(x)))) \right\} \\
&= \sum_{i=0}^{m-1} \left\{ \sum_{i'=0}^{l_1(\sigma_A^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{i+1}(x)))) \right\} \\
&+ \sum_{i'=0}^{l_1(\sigma_A^m(x))-1} f(\sigma_B^{i'}(h(\sigma_A^m(x)))) - \sum_{j'=0}^{k_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x)))) \\
&= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))) \\
&+ \sum_{i'=0}^{l_1(\sigma_A^m(x))-1} f(\sigma_B^{i'}(h(\sigma_A^m(x)))) - \sum_{j'=0}^{k_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x)))) \\
&= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) \\
&- \left\{ \sum_{j'=0}^{l_1(\sigma_A^m(x))+k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))) - \sum_{j'=k_1^m(x)}^{l_1(\sigma_A^m(x))+k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))) \right\} \\
&+ \left\{ \sum_{i'=0}^{l_1(\sigma_A^m(x))+k_1^m(x)-1} f(\sigma_B^{i'}(h(\sigma_A^m(x)))) - \sum_{i'=l_1(\sigma_A^m(x))}^{l_1(\sigma_A^m(x))+k_1^m(x)-1} f(\sigma_B^{i'}(h(\sigma_A^m(x)))) \right\} \\
&- \sum_{j'=0}^{k_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x))))
\end{aligned}$$

The second summand of the first $\{ \cdot \}$ above goes to

$$\begin{aligned}
\sum_{j'=0}^{l_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(\sigma_B^{k_1^m(x)}(h(\sigma_A^m(x)))))) &= \sum_{j'=0}^{l_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(\sigma_B^{l_1^m(x)}(h(x)))) \\
&= \sum_{i'=l_1^m(x)}^{l_1^{m+1}(x)-1} f(\sigma_B^{i'}(h(x))).
\end{aligned}$$

The second summand of the second $\{ \cdot \}$ above goes to

$$\begin{aligned}
\sum_{i'=0}^{k_1^m(x)-1} f(\sigma_B^{i'}(\sigma_B^{l_1(\sigma_A^m(x))}(h(\sigma_A^m(x)))))) &= \sum_{i'=0}^{k_1^m(x)-1} f(\sigma_B^{i'}(\sigma_B^{k_1(\sigma_A^m(x))}(h(\sigma_A^{m+1}(x)))))) \\
&= \sum_{j'=k_1(\sigma_A^m(x))}^{k_1^{m+1}(x)-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x))))).
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \sum_{i=0}^m \left\{ \sum_{i'=0}^{l_1(\sigma_A^i(x))-1} f(\sigma_B^{i'}(h(\sigma_A^i(x)))) - \sum_{j'=0}^{k_1(\sigma_A^i(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{i+1}(x)))) \right\} \\
&= \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) + \sum_{i'=l_1^m(x)}^{l_1^{m+1}(x)-1} f(\sigma_B^{i'}(h(x))) \\
&\quad - \sum_{j'=k_1(\sigma_A^m(x))}^{k_1^{m+1}(x)-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x)))) - \sum_{j'=0}^{k_1(\sigma_A^m(x))-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x)))) \\
&= \sum_{i'=0}^{l_1^{m+1}(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^{m+1}(x)-1} f(\sigma_B^{j'}(h(\sigma_A^{m+1}(x))))
\end{aligned}$$

which shows that the desired equality for $m+1$ holds. \square

For $g \in C(X_A, \mathbb{Z})$, let us define $\Psi_{h^{-1}}(g) \in C(X_B, \mathbb{Z})$ by substituting h^{-1} for h in (4.2) as follows:

$$\Psi_{h^{-1}}(g)(y) = \sum_{i=0}^{l_2(y)-1} g(\sigma_A^i(h^{-1}(y))) - \sum_{j=0}^{k_2(y)-1} g(\sigma_A^j(h^{-1}(\sigma_B(y)))) \quad y \in X_B. \quad (4.3)$$

In Lemma 4.3, by substituting $h(x)$ (resp. $h(\sigma_A(x))$) for x , σ_B for σ_A , h^{-1} for h , σ_A for σ_B , l_2 for l_1 , k_2 for k_1 , and $m = l_1(x)$ (resp. $m = k_1(x)$), respectively, we have the following lemma (i) (resp. (ii)).

Lemma 4.4. *For $g \in C(X_A, \mathbb{Z})$ and $x \in X_A$, we have*

(i)

$$\begin{aligned}
& \sum_{i=0}^{l_1(x)-1} \Psi_{h^{-1}}(g)(\sigma_B^i(h(x))) \\
&= \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} g(\sigma_A^{i'}(x)) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{l_1(x)}(h(x)))).
\end{aligned}$$

(ii)

$$\begin{aligned}
& \sum_{j=0}^{k_1(x)-1} \Psi_{h^{-1}}(g)(\sigma_B^j(h(\sigma_A(x)))) \\
&= \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} g(\sigma_A^{i'+1}(x)) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{k_1(x)}(h(\sigma_A(x)))))).
\end{aligned}$$

We will prove the following proposition.

Proposition 4.5. $\Psi_h \circ \Psi_{h^{-1}} = \text{id}_{C(X_A, \mathbb{Z})}$ and similarly $\Psi_{h^{-1}} \circ \Psi_h = \text{id}_{C(X_B, \mathbb{Z})}$.

Proof. We put

$$\begin{aligned} k_3(x) &= k_2^{l_1(x)}(h(x)) + l_2^{k_1(x)}(h(\sigma_A(x))), & x \in X_A, \\ l_3(x) &= l_2^{l_1(x)}(h(x)) + k_2^{k_1(x)}(h(\sigma_A(x))), & x \in X_A. \end{aligned}$$

By Lemma 3.3 we have $k_3(x) + 1 = l_3(x)$ so that the identity for $g \in C(X_A, \mathbb{Z})$

$$g(x) = \sum_{i=0}^{l_3(x)-1} g(\sigma_A^i(x)) - \sum_{j=0}^{k_3(x)-1} g(\sigma_A^{j+1}(x)), \quad x \in X_A \quad (4.4)$$

holds. By the above lemma, we have

$$\begin{aligned} & \Psi_h(\Psi_{h^{-1}}(g))(x) \\ &= \sum_{i=0}^{l_1(x)-1} \Psi_{h^{-1}}(g)(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} \Psi_{h^{-1}}(g)(\sigma_B^j(h(\sigma_A(x)))) \\ &= \left\{ \sum_{i'=0}^{l_2^{l_1(x)}(h(x))-1} g(\sigma_A^{i'}(x)) - \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{l_1(x)}(h(x))))) \right\} \\ & \quad - \left\{ \sum_{i'=0}^{l_2^{k_1(x)}(h(\sigma_A(x)))-1} g(\sigma_A^{i'+1}(x)) - \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) \right\} \end{aligned}$$

The first $\{ \cdot \}$ above goes to

$$\sum_{i'=0}^{l_3(x)-1} g(\sigma_A^{i'}(x)) \quad (4.5)$$

$$- \left\{ \sum_{i'=l_2^{l_1(x)}(h(x))}^{l_3(x)-1} g(\sigma_A^{i'}(x)) + \sum_{j'=0}^{k_2^{l_1(x)}(h(x))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{l_1(x)}(h(x))))) \right\}. \quad (4.6)$$

The second $\{ \cdot \}$ above goes to

$$\sum_{i'=0}^{k_3(x)-1} g(\sigma_A^{i'+1}(x)) \quad (4.7)$$

$$- \left\{ \sum_{i'=l_2^{k_1(x)}(h(\sigma_A(x)))}^{k_3(x)-1} g(\sigma_A^{i'+1}(x)) + \sum_{j'=0}^{k_2^{k_1(x)}(h(\sigma_A(x)))-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{k_1(x)}(h(\sigma_A(x))))) \right\}. \quad (4.8)$$

Hence we have

$$\Psi_h(\Psi_{h^{-1}}(g))(x) = \{(4.5) + (4.6)\} - \{(4.7) + (4.8)\}$$

Since $\sigma_A^{l_1^{(x)}(h(x))}(x) = \sigma_A^{k_2^{l_1^{(x)}(h(x))}}(h^{-1}(\sigma_B^{l_1^{(x)}}(h(x))))$, we have

$$\begin{aligned}
& - (4.6) \\
& = \sum_{j'=0}^{k_2^{l_1^{(x)}(h(\sigma_A(x)))}-1} g(\sigma_A^{j'}(\sigma_A^{l_1^{(x)}(h(x))}(x))) + \sum_{j'=0}^{k_2^{l_1^{(x)}(h(x))}-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{l_1^{(x)}}(h(x))))) \\
& = \sum_{j'=0}^{k_2^{l_1^{(x)}(h(x))}+k_2^{k_1^{(x)}(h(\sigma_A(x)))}-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{l_1^{(x)}}(h(x))))).
\end{aligned}$$

Since $\sigma_A^{k_1^{(x)}(h(\sigma_A(x)))}(\sigma_A(x)) = \sigma_A^{k_2^{k_1^{(x)}(h(\sigma_A(x)))}}(h^{-1}(\sigma_B^{k_1^{(x)}}(h(\sigma_A(x)))))$, we have

$$\begin{aligned}
& - (4.8) \\
& = \sum_{j'=0}^{k_2^{l_1^{(x)}(h(x))}-1} g(\sigma_A^{j'+l_2^{k_1^{(x)}(h(\sigma_A(x)))}}(\sigma_A(x))) + \sum_{j'=0}^{k_2^{k_1^{(x)}(h(\sigma_A(x)))}-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{k_1^{(x)}}(h(\sigma_A(x))))) \\
& = \sum_{j'=0}^{k_2^{k_1^{(x)}(h(\sigma_A(x)))+k_2^{l_1^{(x)}(h(x))}-1} g(\sigma_A^{j'}(h^{-1}(\sigma_B^{k_1^{(x)}}(h(\sigma_A(x)))))
\end{aligned}$$

We thus have (4.6) = (4.8) by (2.1) and (4.5) – (4.7) = $g(x)$ by (4.4) so that

$$\Psi_h(\Psi_{h^{-1}}(g))(x) = g(x).$$

Similarly we have $\Psi_h \circ \Psi_{h^{-1}} = \text{id}_{C(X_A, \mathbb{Z})}$. □

Lemma 4.6.

$$(i) \quad \Psi_h(f - f \circ \sigma_B) = f \circ h - f \circ h \circ \sigma_A, \quad f \in C(X_B, \mathbb{Z}).$$

$$(ii) \quad \Psi_{h^{-1}}(g - g \circ \sigma_A) = g \circ h^{-1} - g \circ h^{-1} \circ \sigma_B, \quad g \in C(X_A, \mathbb{Z}).$$

Proof. (i) For $f \in C(X_B, \mathbb{Z})$ and $x \in X_A$, we have

$$\begin{aligned}
& \Psi_h(f - f \circ \sigma_B)(x) \\
& = \sum_{i=0}^{l_1(x)-1} (f - f \circ \sigma_B)(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} (f - f \circ \sigma_B)(\sigma_B^j(h(\sigma_A(x)))) \\
& = f(h(x)) - f(\sigma_B^{l_1(x)}(h(x))) - f(h(\sigma_A(x))) + f(\sigma_B^{k_1(x)}(h(\sigma_A(x)))) \\
& = f(h(x)) - f(h(\sigma_A(x))).
\end{aligned}$$

(ii) is similarly shown. □

5 Ordered cohomology groups

Let us denote by

$$\bar{X}_A = \{(x_n)_{n \in \mathbb{Z}} \in \{1, \dots, N\}^{\mathbb{Z}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}$$

the shift space of the two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ for A with shift transformation $\bar{\sigma}_A$ on \bar{X}_A defined by $\bar{\sigma}_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ which is a homeomorphism on \bar{X}_A . Set

$$\bar{H}^A = C(\bar{X}_A, \mathbb{Z}) / \{\xi - \xi \circ \bar{\sigma}_A \mid \xi \in C(\bar{X}_A, \mathbb{Z})\}.$$

The equivalence class of a function $\xi \in C(\bar{X}_A, \mathbb{Z})$ in \bar{H}^A is written $[\xi]$. We define the positive cone \bar{H}_+^A by

$$\bar{H}_+^A = \{[\xi] \in \bar{H}^A \mid \xi(x) \geq 0 \quad \forall x \in \bar{X}_A\}.$$

The pair (\bar{H}^A, \bar{H}_+^A) is called the ordered cohomology group of $(\bar{X}_A, \bar{\sigma}_A)$ (see [1, Section 1.3], [13]). M. Boyle and D. Handelman have proved that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if the ordered cohomology groups (\bar{H}^A, \bar{H}_+^A) and (\bar{H}^B, \bar{H}_+^B) are isomorphic ([1, Theorem 1.12]).

In the same way as above, the ordered group (H^A, H_+^A) for the one-sided topological Markov shift (X_A, σ_A) has been introduced in [8] by setting

$$H^A = C(X_A, \mathbb{Z}) / \{\xi - \xi \circ \sigma_A \mid \xi \in C(X_A, \mathbb{Z})\}$$

and

$$H_+^A = \{[\xi] \in H^A \mid \xi(x) \geq 0 \quad \forall x \in X_A\}.$$

It has been proved that the ordered groups (\bar{H}^A, \bar{H}_+^A) and (H^A, H_+^A) are actually isomorphic ([8, Lemma 3.1]).

By Proposition 4.5 and Lemma 4.6, we see

Proposition 5.1. *Let $h : X_A \rightarrow X_B$ be a homeomorphism which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . Then $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ induces an isomorphism $\bar{\Psi}_h : H^B \rightarrow H^A$ of abelian groups in a natural way.*

In [8], it has been proved that (H^A, H_+^A) is isomorphic to (H^B, H_+^B) as ordered groups by groupoid technique. In this section, we will prove that the above isomorphism Ψ_h preserves their positive cone, that is $\bar{\Psi}_h(H_+^B) = H_+^A$ without groupoid technique so that $\bar{\Psi}_h$ induces an isomorphism from (H^B, H_+^B) to (H^A, H_+^A) as ordered groups.

A subset $S \subset X_A$ is said to be σ_A -invariant if $\sigma_A(S) = S$. We similarly say $\bar{S} \subset \bar{X}_A$ to be $\bar{\sigma}_A$ -invariant if $\bar{\sigma}_A(\bar{S}) = \bar{S}$. We note that a finite subset $S \subset X_A$ is σ_A -invariant if and only if there exists a finite family of periodic points $x(i), i = 1, \dots, m$ such that $x(i)$ is p_i -periodic for some $p_i \in \mathbb{N}$ and

$$S = \{\sigma_A^j(x(i)) \in X_A \mid j = 0, 1, \dots, p_i - 1, i = 1, \dots, m\}.$$

Lemma 5.2 ([8, Lemma 3.2]). *For $f \in C(X_A, \mathbb{Z})$, we have $[f]$ belongs to H_+^A if and only if for every eventually periodic point x with $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s > 0$, the value*

$$\omega_f^{r,s}(x) = \sum_{i=0}^{r-1} f(\sigma_A^i(x)) - \sum_{j=0}^{s-1} f(\sigma_A^j(x)) \quad (5.1)$$

satisfies $\omega_f^{r,s}(x) \geq 0$. If in particular $[f]$ is an order unit of (H^A, H_+^A) if and only if $\omega_f^{r,s}(x) > 0$.

Proof. For $f \in C(X_A, \mathbb{Z})$, it has been shown in [8, Lemma 3.2] that $[f]$ belongs to H_+^A if and only if $\sum_{x \in O} f(x) \geq 0$ for every finite σ_A -invariant set O of X_A . Let x be an eventually periodic point such that $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s > 0$. Let p be the least period of $\sigma_A^s(x)$ so that $r - s = np$ for some $n \in \mathbb{N}$. It then follows that

$$\omega_f^{r,s}(x) = \sum_{i=s}^{r-1} f(\sigma_A^i(x)) = n\{f(\sigma_A^s(x)) + f(\sigma_A^{s+1}(x)) + \cdots + f(\sigma_A^{s+p-1}(x))\}.$$

Since the set $O = \{\sigma_A^s(x), \sigma_A^{s+1}(x), \dots, \sigma_A^{s+p-1}(x)\}$ is a finite σ_A -invariant set of X_A , one sees that $[f] \in H_+^A$ if and only if $\omega_f^{r,s}(x) \geq 0$ by [8, Lemma 3.2]. We know that, by [1, Proposition 3.13], the class $[f]$ is an order unit of (H^A, H_+^A) if and only if $\omega_f^{r,s}(x) > 0$. \square

Lemma 5.3. For $x \in X_A$ with $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s = q \in \mathbb{N}$, put $z = \sigma_B^{l_1^s(x) + k_1^s(x)}(h(x)) \in X_B$ and $r' = l_1^q(\sigma_A^s(x)), s' = k_1^q(\sigma_A^s(x))$. Then we have

$$\sigma_B^{r'}(z) = \sigma_B^{s'}(z), \quad r' \neq s', \quad (5.2)$$

$$\omega_{\Psi_h(f)}^{r,s}(x) = \omega_f^{r',s'}(z) \quad \text{for } f \in C(X_B, \mathbb{Z}). \quad (5.3)$$

Proof. As $l_1^r(x) = l_1^s(x) + r'$ and $k_1^r(x) = k_1^s(x) + s'$, we have

$$\begin{aligned} \sigma_B^{r'}(z) &= \sigma_B^{k_1^s(x)}(\sigma_B^{l_1^r(x)}(h(x))) \\ &= \sigma_B^{k_1^s(x)}(\sigma_B^{k_1^r(x)}(h(\sigma_A^r(x)))) \\ &= \sigma_B^{k_1^r(x)}(\sigma_B^{k_1^s(x)}(h(\sigma_A^s(x)))) \\ &= \sigma_B^{k_1^r(x)}(\sigma_B^{l_1^s(x)}(h(x))) \\ &= \sigma_B^{s'}(z). \end{aligned}$$

The identity (i) of Lemma 3.3 implies that

$$k_2^{r'}(h(\sigma_A^s(x))) + l_2^{s'}(h(\sigma_A^q(\sigma_A^s(x)))) + q = k_2^{s'}(h(\sigma_A^q(\sigma_A^s(x)))) + l_2^{r'}(h(\sigma_A^s(x))).$$

As $\sigma_A^q(\sigma_A^s(x)) = \sigma_A^r(x) = \sigma_A^s(x)$ and $q \neq 0$, we have $r' \neq s'$.

For $f \in C(X_B, \mathbb{Z})$, Lemma 4.3 yields

$$\sum_{i=0}^{m-1} \Psi_h(f)(\sigma_A^i(x)) = \sum_{i'=0}^{l_1^m(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^m(x)-1} f(\sigma_B^{j'}(h(\sigma_A^m(x)))).$$

Hence we have for $m = r, s$

$$\begin{aligned}\omega_{\Psi_h(f)}^{r,s}(x) &= \sum_{i=0}^{r-1} \Psi_h(f)(\sigma_A^i(x)) - \sum_{j=0}^{s-1} \Psi_h(f)(\sigma_A^j(x)) \\ &= \left\{ \sum_{i'=0}^{l_1^r(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^r(x)-1} f(\sigma_B^{j'}(h(\sigma_A^r(x)))) \right\} \\ &\quad - \left\{ \sum_{i'=0}^{l_1^s(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{j'=0}^{k_1^s(x)-1} f(\sigma_B^{j'}(h(\sigma_A^s(x)))) \right\}.\end{aligned}$$

The first summand of the first $\{\cdot\}$ above goes to

$$\sum_{i'=0}^{l_1^r(x)+k_1^s(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{i'=l_1^r(x)}^{l_1^r(x)+k_1^s(x)-1} f(\sigma_B^{i'}(h(x))).$$

The first summand of the second $\{\cdot\}$ above goes to

$$\sum_{i'=0}^{l_1^s(x)+k_1^r(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{i'=l_1^s(x)}^{l_1^s(x)+k_1^r(x)-1} f(\sigma_B^{i'}(h(x))).$$

Hence we have

$$\begin{aligned}\omega_{\Psi_h(f)}^{r,s}(x) &= \left\{ \sum_{i'=0}^{l_1^r(x)+k_1^s(x)-1} f(\sigma_B^{i'}(h(x))) - \sum_{i'=0}^{l_1^s(x)+k_1^r(x)-1} f(\sigma_B^{i'}(h(x))) \right\} \\ &\quad - \left\{ \sum_{i'=l_1^r(x)}^{l_1^r(x)+k_1^s(x)-1} f(\sigma_B^{i'}(h(x))) + \sum_{j'=0}^{k_1^r(x)-1} f(\sigma_B^{j'}(h(\sigma_A^r(x)))) \right\} \\ &\quad + \left\{ \sum_{i'=l_1^s(x)}^{l_1^s(x)+k_1^r(x)-1} f(\sigma_B^{i'}(h(x))) + \sum_{j'=0}^{k_1^s(x)-1} f(\sigma_B^{j'}(h(\sigma_A^s(x)))) \right\}.\end{aligned}$$

Since $l_1^r(x) = l_1^s(x) + r'$ and $k_1^r(x) = k_1^s(x) + s'$, the first $\{\cdot\}$ above goes to

$$\sum_{i=0}^{r'-1} f(\sigma_B^i(z)) - \sum_{j=0}^{s'-1} f(\sigma_B^j(z)). \quad (5.4)$$

Since $\sigma_B^{l_1^r(x)}(h(x)) = \sigma_B^{k_1^r(x)}(h(\sigma_A^r(x)))$, the second $\{\cdot\}$ above goes to

$$\sum_{j'=0}^{k_1^r(x)+k_1^s(x)-1} f(\sigma_B^{j'}(h(\sigma_A^r(x)))). \quad (5.5)$$

Since $\sigma_B^{l_1^s(x)}(h(x)) = \sigma_B^{k_1^s(x)}(h(\sigma_A^s(x)))$, the third $\{\cdot\}$ above goes to

$$\sum_{j'=0}^{k_1^s(x)+k_1^r(x)-1} f(\sigma_B^{j'}(h(\sigma_A^s(x)))). \quad (5.6)$$

As $\sigma_A^r(x) = \sigma_A^s(x)$, we have (5.5) = (5.6), so that $\omega_{\Psi_h(f)}^{r,s}(x) = (5.4)$. \square

We define for $n = 1, 2, \dots$

$$\begin{aligned} c_1(x) &= l_1(x) - k_1(x), & c_1^n(x) &= l_1^n(x) - k_1^n(x), & x &\in X_A, \\ c_2(y) &= l_2(y) - k_2(y), & c_2^n(y) &= l_2^n(y) - k_2^n(y), & y &\in X_B. \end{aligned}$$

The function c_1 (resp. c_2) is called the cocycle function for h (resp. h^{-1}). It is clear that the following cocycle conditions hold:

$$\begin{aligned} c_1^{n+m}(x) &= c_1^n(x) + c_1^m(\sigma_A^n(x)), & n, m &\in \mathbb{N}, x \in X_A \\ c_2^{n+m}(y) &= c_2^n(y) + c_2^m(\sigma_B^n(y)), & n, m &\in \mathbb{N}, y \in X_B. \end{aligned}$$

Lemma 5.4. *The following conditions are equivalent:*

- (i) $[\Psi_h(f)] \in H_+^A$ for every $f \in C(X_B, \mathbb{Z})$ with $[f] \in H_+^B$.
- (ii) $[c_1] \in H_+^A$.
- (iii) $\omega_{c_1}^{r,s}(x) > 0$ for $x \in X_A$ such that $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s > 0$.
- (iv) $c_1^q(\sigma_A^s(x)) > 0$ for $x \in X_A$ such that $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s = q > 0$.
- (v) $l_1^r(x) + k_1^s(x) > k_1^r(x) + l_1^s(x)$ for $x \in X_A$ such that $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s > 0$.
- (vi) $r' > s'$ for $x \in X_A$ such that $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s = q > 0$, where $r' = l_1^q(\sigma_A^s(x))$, $s' = k_1^q(\sigma_A^s(x))$.

Proof. Let $x \in X_A$ satisfy $\sigma_A^r(x) = \sigma_A^s(x)$ for some $r, s \in \mathbb{Z}_+$ such that $r - s = q \in \mathbb{N}$. We then note that $r' - s' \neq 0$ by Lemma 5.3. The equivalences among (iii), (iv), (v) and (vi) come from the following equalities:

$$\begin{aligned} \omega_{c_1}^{r,s}(x) &= c_1^r(x) - c_1^s(x) \\ &= (l_1^r(x) - l_1^s(x)) - (k_1^r(x) - k_1^s(x)) \\ &= \sum_{i=0}^{q-1} l_1(\sigma_A^i(\sigma_A^s(x))) - \sum_{i=0}^{q-1} k_1(\sigma_A^i(\sigma_A^s(x))) \\ &= r' - s' = c_1^q(\sigma_A^s(x)). \end{aligned}$$

The equivalence between (ii) and (iii) follows from Lemma 5.2. Suppose that the condition (i) holds. Take the constant function $1_B(y) = 1, y \in X_B$ as a function $f \in C(X_B, \mathbb{Z})$. The condition (i) implies that $[\Psi_h(1_B)] \in H_+^A$. For $x \in X_A$ we have @

$$\Psi_h(1_B)(x) = \sum_{i=0}^{l_1(x)-1} 1_B(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)-1} 1_B(\sigma_B^j(h(\sigma_A(x)))) = c_1(x)$$

so that $[c_1] \in H_+^A$ and the condition (ii) holds. We finally assume the condition (vi). For a function $f \in C(X_B, \mathbb{Z})$ with $[f] \in H_+^B$ and $x \in X_A$ with $\sigma_A^r(x) = \sigma_A^s(x)$ and $r - s > 0$, the condition (vi) implies $\omega_f^{r',s'}(z) > 0$ from $[f] \in H_+^B$ by Lemma 5.2, where $z = \sigma_B^{l_1^s(x)+k_1^s(x)}(h(x))$. Hence the equality (5.3) implies $\omega_{\Psi_h(f)}^{r,s}(x) > 0$ so that $[\Psi_h(f)] \in H_+^A$ by Lemma 5.2 again. This implies the condition (i). \square

In the rest of the section, we will show that $[c_1] \in H_+^A$ always holds.

Definition 5.5. For $r, s \in \mathbb{Z}_+$, an eventually periodic point $x \in X_A$ is said to be (r, s) -attracting if it satisfies the following two conditions:

- (i) $\sigma_A^r(x) = \sigma_A^s(x)$.
- (ii) For any clopen neighborhood $W \subset X_A$ of x , there exist clopen sets $U, V \subset X_A$ and a homeomorphism $\varphi : V \rightarrow U$ such that
 - (a) $x \in U \subset V \subset W$.
 - (b) $\varphi(x) = x$.
 - (c) $\sigma_A^r(\varphi(w)) = \sigma_A^s(w)$ for all $w \in V$.
 - (d) $\lim_{n \rightarrow \infty} \varphi^n(w) = x$ for all $w \in V$.

For a word $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$, denote by $U_\mu \subset X_A$ the cylinder set

$$U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}.$$

Lemma 5.6. *An eventually periodic point is (r, s) -attracting for some $r, s \in \mathbb{Z}_+$.*

Proof. Let $x \in X_A$ be an eventually periodic point such that $\sigma_A^r(x) = \sigma_A^s(x)$ with $r \neq s$. We may assume that $r > s$. Put the words $\nu = x_{[1, s]}$, $\xi = x_{[s+1, r]}$, $\mu = x_{[1, r]}$. One has $\mu = \nu\xi$ and $x = \nu\xi\xi\xi \cdots$. For a clopen neighborhood $W \subset X_A$ of x , there exist $L \in \mathbb{N}$ such that by putting $\bar{\nu} = \nu \overbrace{\xi \cdots \xi}^L$ and $\bar{\mu} = \mu \overbrace{\xi \cdots \xi}^{L+1}$, one has $x \in U_{\bar{\mu}} \subset U_{\bar{\nu}} \subset W$. We set $V = U_{\bar{\nu}}$ and $U = U_{\bar{\mu}}$. Define $\varphi : V \rightarrow U$ by substituting $\bar{\mu}$ for the left most word $\bar{\nu}$ of elements of V . It is a homeomorphism from V to U such that $\varphi(x) = x$. Since $|\nu| = s, |\mu| = r$, the equalities $\sigma_A^r(\varphi(w)) = \sigma_A^s(w)$ for all $w \in V$ hold. As $\varphi^n(w)$ begins with $\nu \overbrace{\xi \cdots \xi}^{L+n}$, we have $\lim_{n \rightarrow \infty} \varphi^n(w) = x$ for all $w \in V$. \square

Lemma 5.7. *If an eventually periodic point is (r, s) -attracting, then $r > s$.*

Proof. Let $x \in X_A$ be (r, s) -attracting. For $W = X_A$, take clopen sets $U, V \subset X_A$ and a homeomorphism $\varphi : V \rightarrow U$ satisfying the conditions (ii) of Definition 5.5. We note that the matrix A satisfies condition (I) in the sense of [2] so that X_A is homeomorphic to a Cantor set. Assume that $r \leq s$. We have two cases.

Case 1 : $r = s$.

Take $w \in V$ such that $w_{[r+1, \infty)} \neq x_{[r+1, \infty)}$. By the condition (c) of (ii) in Definition 5.5, one sees $\sigma_A^r(\varphi^n(w)) = \sigma_A^r(w), n \in \mathbb{N}$ so that $\lim_{n \rightarrow \infty} \sigma_A^r(\varphi^n(w)) = \sigma_A^r(w)$, which contradicts to the condition $\lim_{n \rightarrow \infty} \varphi^n(w) = x$ with $w_{[r+1, \infty)} \neq x_{[r+1, \infty)}$.

Case 2 : $r < s$.

Put $q = s - r \in \mathbb{N}$. For all $w \in V$, we have $\varphi(w)_{[r+1, \infty)} = w_{[s+1, \infty)}$. As $\varphi^n(w) \in V$ for $n \in \mathbb{N}$, we have $\varphi^n(w)_{[r+1, \infty)} = w_{[s+(n-1)q+1, \infty)}$ so that $\varphi^n(w)_{[r+1, r+q]} = w_{[s+(n-1)q+1, s+nq]}$ for all $n \in \mathbb{N}$. Hence $\lim_{n \rightarrow \infty} \varphi^n(w)$ does not exist unless $\sigma_A^r(w)$ is q -periodic. There exists a point $w \in V$ which is not an eventually periodic point, a contradiction.

Therefore the above two cases do not occur. \square

Lemma 5.8. *If x is (r, s) -attracting, then $h(x)$ is $(l_1^r(x) + k_1^s(x), k_1^r(x) + l_1^s(x))$ -attracting.*

Proof. For a clopen neighborhood $W' \subset X_B$ of $h(x)$, put a clopen neighborhood $W = h^{-1}(W') \subset X_A$ of x . Since the functions $l_1^r, k_1^s, k_1^r, l_1^s$ are all continuous, one may take W small enough such that

$$l_1^r(w) = l_1^r(x), \quad k_1^s(w) = k_1^s(x), \quad k_1^r(w) = k_1^r(x), \quad l_1^s(w) = l_1^s(x)$$

for all $w \in W$. Put $r' = l_1^r(x) + k_1^s(x), s' = k_1^r(x) + l_1^s(x)$. By Lemma 3.2, one has $\sigma_B^{r'}(h(x)) = \sigma_B^{s'}(h(x))$. Take clopen sets $U, V \subset X_A$ and a homeomorphism $\varphi : V \rightarrow U$ satisfying the condition (ii) of Definition 5.5. We set $U' = h(U), V' = h(V)$ of X_B and a homeomorphism $\varphi' = h \circ \varphi \circ h^{-1}|_{V'} : V' \rightarrow U'$. They satisfy $h(x) \in U' \subset V' \subset W'$. As $\sigma_A^r(\varphi(w)) = \sigma_A^s(w)$ for $w \in V$, Lemma 3.2 ensures us

$$\sigma_B^{l_1^r(\varphi(w)) + k_1^s(w)}(h(\varphi(w))) = \sigma_B^{k_1^r(\varphi(w)) + l_1^s(w)}(h(w)) \quad (5.7)$$

for $w \in V$. Since $\varphi(w) \in V$ for $w \in V$, one sees that $l_1^r(\varphi(w)) = l_1^r(x), k_1^s(w) = k_1^s(x), k_1^r(\varphi(w)) = k_1^r(x), l_1^s(w) = l_1^s(x)$. Hence the equality (5.7) goes to

$$\sigma_B^{r'}(h(\varphi(w))) = \sigma_B^{s'}(h(w)) \quad \text{and hence} \quad \sigma_B^{r'}(\varphi'(h(w))) = \sigma_B^{s'}(h(w))$$

for all $w \in V$. The equality $\lim_{n \rightarrow \infty} \varphi'^n(w) = h(x)$ for $w \in V$ is easily verified, so that $h(x)$ is (r', s') -attracting. \square

Corollary 5.9. *Keep the above notations. If $x \in X_A$ satisfies $\sigma_A^r(x) = \sigma_A^s(x)$ for some $r > s$, then $l_1^r(x) + k_1^s(x) > k_1^r(x) + l_1^s(x)$ and hence $c_1^q(\sigma_A^s(x)) > 0$ where $q = r - s$.*

By using the above corollary with Lemma 5.4, we reach the following proposition and theorem.

Proposition 5.10. *The class $[c_1]$ of the cocycle function $c_1(x) = l_1(x) - k_1(x)$ for $x \in X_A$ in H^A gives rise to a positive element in the ordered cohomology group (H^A, H_+^A) which is an order unit in (H^A, H_+^A) .*

Therefore we have

Theorem 5.11 (cf. [8, Theorem 3.5]). *Let h be a homeomorphism from X_A to X_B which gives rise to a continuous orbit equivalence between the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) . Then there exist isomorphisms $\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z})$ and $\Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$ which are inverses to each other such that*

- (i) $\Psi_h(1_B)(x) = l_1(x) - k_1(x)$ for $x \in X_A$,
- (ii) $\Psi_{h^{-1}}(1_A)(y) = l_2(y) - k_2(y)$ for $y \in X_B$,
- (iii) $[\Psi_h(f)] \in H_+^A$ for $[f] \in H_+^B$,
- (iv) $[\Psi_{h^{-1}}(g)] \in H_+^B$ for $[g] \in H_+^A$

so that Ψ_h induces an isomorphism $\bar{\Psi}_h : (H^B, H_+^B) \rightarrow (H^A, H_+^A)$ of the ordered cohomology groups (H^A, H_+^A) and (H^B, H_+^B) as ordered groups.

6 Periodic points and zeta functions

Continuous orbit equivalence between one-sided topological Markov shifts preserves their eventually periodic points. Eventually periodic points of a one-sided topological Markov shift naturally yield periodic points of the two-sided topological Markov shift. In this section, we will study periodic points of two-sided topological Markov shifts whose one-sided topological Markov shifts are continuously orbit equivalent. Recall that $(\bar{X}_A, \bar{\sigma}_A)$ stands for the two-sided topological Markov shift for matrix A . For $p \in \mathbb{N}$, put the set of periodic points

$$\text{Per}_p(\bar{X}_A) = \{\bar{x} = (x_n)_{n \in \mathbb{Z}} \in \bar{X}_A \mid \bar{\sigma}_A^p(\bar{x}) = \bar{x}\}$$

and $\text{Per}_*(\bar{X}_A) = \bigcup_{p=1}^{\infty} \text{Per}_p(\bar{X}_A)$. For $\bar{x} \in \text{Per}_*(\bar{X}_A)$, the subset $\gamma = \{\bar{\sigma}_A^n(\bar{x}) \in \bar{X}_A \mid n \in \mathbb{Z}\}$ of \bar{X}_A is called the periodic orbit of \bar{x} under $\bar{\sigma}_A$. We call the cardinality $|\gamma|$ of γ the period of γ which is the least period of \bar{x} under $\bar{\sigma}_A$. If $\bar{x} \in \text{Per}_p(\bar{X}_A)$, then $p = k|\gamma|$ for some $k \in \mathbb{N}$. Let $P_{orb}(\bar{X}_A)$ be the set of periodic orbits of $(\bar{X}_A, \bar{\sigma}_A)$. Denote by $\pi_A : \bar{X}_A \rightarrow X_A$ the restriction of \bar{X}_A to X_A defined by $\pi_A((x_n)_{n \in \mathbb{Z}}) = (x_n)_{n \in \mathbb{N}}$. We are assuming that h is a homeomorphism from X_A to X_B which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . Recall that $c_1^p \in C(X_A, \mathbb{Z})$ for $p \in \mathbb{N}$ is the cocycle function defined by $c_1^p(x) = l_1^p(x) - k_1^p(x)$, $x \in X_A$.

Lemma 6.1. *There exists a map $\psi_h : \text{Per}_*(\bar{X}_A) \rightarrow \text{Per}_*(\bar{X}_B)$ such that for $\bar{x} \in \text{Per}_p(\bar{X}_A)$,*

$$\sigma_B^{k_1^p(x)}(\pi_B(\psi_h(\bar{x}))) = \sigma_B^{k_1^p(x)}(h(\pi_A(\bar{x}))), \quad (6.1)$$

$$\bar{\sigma}_B^{k_1(x)}(\psi_h(\bar{\sigma}_A(\bar{x}))) = \bar{\sigma}_B^{l_1(x)}(\psi_h(\bar{x})), \quad (6.2)$$

$$\bar{\sigma}_B^{c_1^p(x)}(\psi_h(\bar{x})) = \psi_h(\bar{x}), \quad (6.3)$$

where $x = \pi_A(\bar{x})$.

Proof. For $\bar{x} \in \text{Per}_p(\bar{X}_A)$, put $x = \pi_A(\bar{x})$ which is a p -periodic point of X_A . By (3.10) and Corollary 5.9, $h(x)$ is an eventually periodic point of X_B such that $\sigma_B^{k_1^p(x)}(h(x))$ is a $c_1^p(x)$ -periodic point. There exists a unique element \bar{y} in \bar{X}_B satisfying $\bar{\sigma}_B^{c_1^p(x)}(\bar{y}) = \bar{y}$ and $\pi_B(\bar{\sigma}_B^{k_1^p(x)}(\bar{y})) = \sigma_B^{k_1^p(x)}(h(x))$. That is a unique extension of the $c_1^p(x)$ -periodic point $\sigma_B^{k_1^p(x)}(h(x))$ to a two-sided sequence in \bar{X}_B . Define $\psi_h(\bar{x}) = \bar{y} \in \bar{X}_B$ so that

$$\psi_h(\bar{x})_{[k_1^p(\sigma_A(x)), \infty)} = h(\pi_A(\bar{x}))_{[k_1^p(\sigma_A(x)), \infty)}, \quad (6.4)$$

and hence the equalities (6.1) and (6.3) are obvious. By $k_1^p(\sigma_A(x)) = k_1^p(x)$ and (6.4), we have

$$\begin{aligned} \sigma_B^{k_1(x)}(h(\sigma_A(x)))_{[k_1^p(\sigma_A(x)), \infty)} &= h(\pi_A(\bar{\sigma}_A(\bar{x})))_{[k_1^p(\sigma_A(x)) + k_1(x), \infty)} \\ &= \psi_h(\bar{\sigma}_A(\bar{x}))_{[k_1^p(\sigma_A(x)) + k_1(x), \infty)} \\ &= \bar{\sigma}_B^{k_1(x)}(\psi_h(\bar{\sigma}_A(\bar{x})))_{[k_1^p(x), \infty)} \end{aligned}$$

and

$$\begin{aligned}
\sigma_B^{l_1(x)}(h(x))_{[k_1^p(\sigma_A(x)), \infty)} &= h(\pi_A(\bar{x}))_{[k_1^p(\sigma_A(x)) + l_1(x), \infty)} \\
&= \psi_h(\bar{x})_{[k_1^p(x) + l_1(x), \infty)} \\
&= \bar{\sigma}_B^{l_1(x)}(\psi_h(\bar{x}))_{[k_1^p(x), \infty)}
\end{aligned}$$

so that the identity (2.1) implies

$$\bar{\sigma}_B^{k_1(x)}(\psi_h(\bar{\sigma}_A(\bar{x})))_{[k_1^p(x), \infty)} = \bar{\sigma}_B^{l_1(x)}(\psi_h(\bar{x}))_{[k_1^p(x), \infty)}$$

As both $\bar{\sigma}_B^{k_1(x)}(\psi_h(\bar{\sigma}_A(\bar{x})))$ and $\bar{\sigma}_B^{l_1(x)}(\psi_h(\bar{x}))$ are periodic, we obtain (6.2). Thus $\psi_h : \text{Per}_*(\bar{X}_A) \longrightarrow \text{Per}_*(\bar{X}_B)$ satisfies the desired properties. \square

By (6.2), the above map $\psi_h : \text{Per}_*(\bar{X}_A) \longrightarrow \text{Per}_*(\bar{X}_B)$ preserves each orbit of periodic points so that it induces a map

$$\xi_h : P_{orb}(\bar{X}_A) \longrightarrow P_{orb}(\bar{X}_B)$$

such that $\xi_h(\gamma) = \{\bar{\sigma}_B^m(\psi_h(\bar{\sigma}_A^n(\bar{x}))) \mid n, m \in \mathbb{Z}\} \subset \bar{X}_B$ for $\gamma = \{\bar{\sigma}_A^n(\bar{x}) \mid n \in \mathbb{Z}\} \subset \bar{X}_A$. We also have a map $\psi_{h^{-1}} : \text{Per}_*(\bar{X}_B) \longrightarrow \text{Per}_*(\bar{X}_A)$ and the induced map $\xi_{h^{-1}} : P_{orb}(\bar{X}_B) \longrightarrow P_{orb}(\bar{X}_A)$ for the inverse $h^{-1} : X_B \longrightarrow X_A$ of h .

Lemma 6.2. *For $\bar{x} \in \text{Per}_p(\bar{X}_A)$, put $x = \pi_A(\bar{x})$, $q = c_1^p(x)$, $n = k_1^p(x)$, $\bar{y} = \psi_h(\bar{x})$, $y = \pi_B(\bar{y})$. Then we have*

$$\sigma_A^{l_2^n(y) + k_2^q(y)}(\pi_A(\psi_{h^{-1}}(\psi_h(\bar{x})))) = \sigma_A^{l_2^n(y) + k_2^q(y)}(\pi_A(\bar{x}))$$

so that $\xi_{h^{-1}} \circ \xi_h = \text{id}$ on $P_{orb}(\bar{X}_A)$, and similarly $\xi_h \circ \xi_{h^{-1}} = \text{id}$ on $P_{orb}(\bar{X}_B)$.

Proof. By (6.1) for h^{-1} and $\bar{y} = \psi_h(\bar{x})$, we have

$$\sigma_A^{k_2^q(y)}(\pi_A(\psi_{h^{-1}}(\psi_h(\bar{x})))) = \sigma_A^{k_2^q(y)}(h^{-1}(\pi_B(\psi_h(\bar{x})))).$$

As in (6.1), we have $\sigma_B^n(\pi_B(\psi_h(\bar{x}))) = \sigma_B^n(h(x))$. As

$$\begin{aligned}
\sigma_A^{l_2^n(y)}(h^{-1}(\pi_B(\psi_h(\bar{x})))) &= \sigma_A^{k_2^n(y)}(h^{-1}(\sigma_B^n(\pi_B(\psi_h(\bar{x})))) \\
&= \sigma_A^{k_2^n(y)}(h^{-1}(\sigma_B^n(h(x)))) = \sigma_A^{l_2^n(y)}(\pi_A(\bar{x})),
\end{aligned}$$

we see that

$$\begin{aligned}
&\sigma_A^{l_2^n(y) + k_2^q(y)}(\pi_A(\psi_{h^{-1}}(\psi_h(\bar{x})))) \\
&= \sigma_A^{k_2^q(y)}(\sigma_A^{l_2^n(y)}(h^{-1}(\pi_B(\psi_h(\bar{x})))) = \sigma_A^{l_2^n(y) + k_2^q(y)}(\pi_A(\bar{x})).
\end{aligned}$$

Hence $\pi_A(\psi_{h^{-1}}(\psi_h(\bar{x})))$ and $\pi_A(\bar{x})$ are in the same orbit in X_A so that $\psi_{h^{-1}}(\psi_h(\bar{x}))$ and \bar{x} are in the same orbit in \bar{X}_A . Therefore we see that $\xi_{h^{-1}} \circ \xi_h = \text{id}$ on $P_{orb}(\bar{X}_A)$ and similarly $\xi_h \circ \xi_{h^{-1}} = \text{id}$ on $P_{orb}(\bar{X}_B)$. \square

The above lemma says that continuous orbit equivalence between one-sided topological Markov shifts yields a bijective correspondence between the sets of periodic orbits of their two-sided topological Markov shifts.

Lemma 6.3. *Let $x \in X_A$ satisfy $\sigma_A^r(x) = \sigma_A^s(x)$ such that $r - s = q = np \in \mathbb{N}$ for some $n \in \mathbb{N}$, where p is the least period of $\sigma_A^s(x)$. Then*

$$c_1^q(\sigma_A^s(x)) = n \cdot c_1^p(\sigma_A^s(x)).$$

Proof. As $\sigma_A^{s+j}(x) = \sigma_A^{s+ip+j}(x)$ for $j = 0, 1, \dots, p-1$ and $i = 0, 1, \dots, n-1$, we have

$$c_1^q(\sigma_A^s(x)) = \sum_{m=0}^{q-1} c_1(\sigma_A^{s+m}(x)) = n \sum_{j=0}^{p-1} c_1(\sigma_A^{s+j}(x)) = n \cdot c_1^p(\sigma_A^s(x)).$$

□

Lemma 6.4. *Let $x \in X_A$ satisfy $\sigma_A^r(x) = \sigma_A^s(x)$ such that $r - s = q \in \mathbb{N}$. Put $z = \sigma_B^{l_1^s(x) + k_1^s(x)}(h(x)) \in X_B$, $r' = l_1^q(\sigma_A^s(x))$, $s' = k_1^q(\sigma_A^s(x))$, $q' = r' - s'$. Then we have*

$$c_2^{q'}(\sigma_B^{s'}(z)) = r - s.$$

Proof. We note that by Corollary 5.9, $q' = l_1^r(x) + k_1^s(x) - (k_1^r(x) + l_1^s(x)) > 0$. It then follows that by Lemma 5.3 and (5.3), (5.4),

$$\begin{aligned} c_2^{q'}(\sigma_B^{s'}(z)) &= l_2^{r'}(z) - l_2^{s'}(z) - (k_2^{r'}(z) - k_2^{s'}(z)) \\ &= \omega_{\Psi_h(l_2)}^{r,s}(x) - \omega_{\Psi_h(k_2)}^{r,s}(x) = \omega_{\Psi_h(c_2)}^{r,s}(x). \end{aligned}$$

As $c_2 = \Psi_{h^{-1}}(1_A)$ and $\Psi_h(\Psi_{h^{-1}}(1_A)) = 1_A$, we have

$$c_2^{q'}(\sigma_B^{s'}(z)) = \omega_{1_A}^{r,s}(x) = r - s.$$

□

Lemma 6.5. *For $\gamma \in P_{orb}(\bar{X}_A)$ with $|\gamma| = p$, take $\bar{x} \in \gamma$ and put $x = \pi_A(\bar{x})$. Then we have*

$$|\xi_h(\gamma)| = c_1^p(x) (= \omega_{c_1}^{r,s}(x)).$$

Proof. The least period of x is p . Put $z = h(x)$ and $r' = l_1^p(x)$, $s' = k_1^p(x)$, $q' = r' - s'$. By Corollary 5.9 for $r = p$ and $s = 0$, we know that $r' - s' = q' = c_1^p(x) > 0$. Denote by p' the least eventual period of z . By Lemma 5.3, z has an eventual period $r' - s' = q'$. Hence we have

$$c_1^p(x) = q' = n' \cdot p' \quad \text{for some } n' \in \mathbb{N}. \quad (6.5)$$

Since $|\xi_h(\gamma)|$ coincides with p' , it suffices to prove that $n' = 1$. As p' is the least period of $\sigma_B^{s'}(z)$, by applying Lemma 6.3 for $\sigma_B^{r'}(z) = \sigma_B^{s'}(z)$, $r' - s' = q' = n'p'$ we have

$$c_2^{q'}(\sigma_B^{s'}(z)) = n' \cdot c_2^{p'}(\sigma_B^{s'}(z)). \quad (6.6)$$

By applying Lemma 6.4 for $r = p, s = 0, z = h(x), r' = l_1^p(x), s' = k_1^p(x), r' - s' = q'$, we have

$$c_2^{q'}(\sigma_B^{s'}(z)) = p. \quad (6.7)$$

Since $\sigma_B^{s'+p'}(z) = \sigma_B^{s'}(z)$, by putting $x' = \sigma_A^{l_2^{s'}(z)+k_2^{s'}(z)}(x)$, we have

$$\begin{aligned} \sigma_A^{l_2^{p'}(\sigma_B^{s'}(z))}(x') &= \sigma_A^{l_2^{p'}(\sigma_B^{s'}(z))+l_2^{s'}(z)+k_2^{s'}(z)}(h^{-1}(z)) \\ &= \sigma_A^{k_2^{s'}(z)}(\sigma_A^{l_2^{s'+p'}(z)}(h^{-1}(z))) \\ &= \sigma_A^{k_2^{s'}(z)}(\sigma_A^{k_2^{s'+p'}(z)}(h^{-1}(\sigma_B^{s'+p'}(z)))) \\ &= \sigma_A^{k_2^{s'+p'}(z)}(\sigma_A^{k_2^{s'}(z)}(h^{-1}(\sigma_B^{s'}(z)))) \\ &= \sigma_A^{k_2^{p'}(\sigma_B^{s'}(z))+k_2^{s'}(z)+l_2^{s'}(z)}(h^{-1}(z))) \\ &= \sigma_A^{k_2^{p'}(\sigma_B^{s'}(z))}(x') \end{aligned}$$

so that

$$\sigma_A^{l_2^{p'}(\sigma_B^{s'}(z))}(x') = \sigma_A^{k_2^{p'}(\sigma_B^{s'}(z))}(x'). \quad (6.8)$$

The least period of x' is equal to that of x which is p . By (6.8), we have

$$c_2^{p'}(\sigma_B^{s'}(z)) = l_2^{p'}(\sigma_B^{s'}(z)) - k_2^{p'}(\sigma_B^{s'}(z)) = m' \cdot p \quad \text{for some } m' \in \mathbb{Z}.$$

By (6.6) and (6.7), we see

$$p = c_2^{q'}(\sigma_B^{s'}(z)) = n' \cdot c_2^{p'}(\sigma_B^{s'}(z)) = n' \cdot m' \cdot p.$$

We thus conclude that $n' = m' = 1$ so that (6.5) implies $c_1^p(x) = p' = |\xi_h(\gamma)|$. \square

For $f \in C(X_A, \mathbb{Z})$ with $[f] \in H_+^A$ and a finite periodic orbit $O \subset X_A$, which has a point $x \in X_A$ and $p \in \mathbb{N}$ such that $x = \sigma_A^p(x)$ and $O = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\}$, we set

$$\beta_O([f]) = \sum_{i=0}^{p-1} f(\sigma_A^i(x)).$$

We may naturally identify finite periodic orbits of (X_A, σ_A) with finite periodic orbits of $(\bar{X}_A, \bar{\sigma}_A)$. If for $\gamma \in P_{orb}(\bar{X}_A)$ and $f = 1_A$, one sees that

$$\beta_\gamma([1_A]) = \sum_{x \in \gamma} 1_A(x) = |\gamma| : \quad \text{the length of periodic orbit of } \gamma.$$

Lemma 6.6. *For a periodic point x in X_A with least period $p \in \mathbb{N}$, put $\gamma = \{x, \sigma_A(x), \dots, \sigma_A^{p-1}(x)\}$ the orbit of x . Then we have*

$$\beta_\gamma([c_1]) = |\xi_h(\gamma)|.$$

Proof. In the preceding lemma, one knows that $|\xi_h(\gamma)| = c_1^p(x)$. Since $\beta_\gamma([c_1]) = \sum_{i=0}^{p-1} c_1(\sigma_A^i(x)) = c_1^p(x)$, one has $\beta_\gamma([c_1]) = |\xi_h(\gamma)|$. \square

Denote by $|\text{Per}_n(\bar{X}_A)|$ the cardinality of the set $\text{Per}_n(\bar{X}_A)$ of n -periodic points of $(\bar{X}, \bar{\sigma}_A)$. The zeta function $\zeta_A(t)$ for $(\bar{X}, \bar{\sigma}_A)$ is defined by

$$\zeta_A(t) = \exp \left(\sum_{n=1}^{\infty} \frac{t^n}{n} |\text{Per}_n(\bar{X}_A)| \right).$$

It has the following Euler product formula (see [5, Section 6.4])

$$\zeta_A(t) = \prod_{\gamma \in P_{orb}(\bar{X}_A)} (1 - t^{|\gamma|})^{-1}.$$

In [1, p. 176], the zeta function of an order unit in ordered cohomology group has been studied related to flow equivalence of topological Markov shifts. In our situation, the class $[c_1]$ of the cocycle function $c_1(x) = l_1(x) - k_1(x)$, $x \in X_A$ gives rise to an order unit in the ordered group (H^A, H_+^A) . Hence the zeta function $\zeta_{[c_1]}(t)$ for the order unit $[c_1]$ may be defined in the sense of [1], which goes to

$$\zeta_{[c_1]}(t) = \prod_{\gamma \in P_{orb}(\bar{X}_A)} (1 - t^{\beta_\gamma([c_1])})^{-1}.$$

We note that by putting $t = e^{-s}$, the zeta function $\zeta_{[c_1]}(t)$ coincides with the following zeta function $\zeta_{\bar{\sigma}_A, \bar{c}_1}(s)$ so called the dynamical zeta function with potential $\bar{c}_1(\bar{x}) = c_1 \circ \pi_A(\bar{x})$, $\bar{x} \in \bar{X}_A$

$$\zeta_{\bar{\sigma}_A, \bar{c}_1}(s) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\bar{x} \in \text{Per}_n(\bar{X}_A)} \exp \left(-s \sum_{k=0}^{n-1} \bar{c}_1(\bar{\sigma}_A^k(\bar{x})) \right) \right\} \quad (6.9)$$

(see [11], [15], [16]). If $[c_1] = [1_A]$ in H_+^A so that the unital ordered groups $(H^A, H_+^A, [1_A])$ and $(H^B, H_+^B, [1_B])$ are isomorphic, then $\beta_\gamma([c_1]) = |\gamma|$ so that

$$\zeta_{[c_1]}(t) = \zeta_A(t).$$

As $\xi_h : P_{orb}(\bar{X}_A) \longrightarrow P_{orb}(\bar{X}_B)$ is bijective, the preceding lemma implies that the equality

$$\zeta_{[c_1]}(t) = \prod_{\gamma \in P_{orb}(\bar{X}_A)} (1 - t^{|\xi_h(\gamma)|})^{-1} = \zeta_B(t)$$

holds. We thus have the following theorem which describes structure of periodic points of the two-sided topological Markov shifts in continuously orbit equivalent one-sided topological Markov shifts.

Theorem 6.7. *Suppose that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent through a homeomorphism h from X_A to X_B satisfying (2.1) and (2.2). Let $\zeta_A(t)$ and $\zeta_B(t)$ be the zeta functions for their two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ respectively. Then the zeta functions $\zeta_A(t)$ and $\zeta_B(t)$ coincide with the zeta functions $\zeta_{[c_2]}(t)$ and $\zeta_{[c_1]}(t)$ for the cocycle functions $c_2(y) = l_2(y) - k_2(y)$, $y \in X_B$ and $c_1(x) = l_1(x) - k_1(x)$, $x \in X_A$ respectively. That is*

$$\zeta_A(t) = \zeta_{[c_2]}(t), \quad \zeta_B(t) = \zeta_{[c_1]}(t).$$

One may easily see that the condition $[c_1] = [1_A]$ in H^A implies $[c_2] = [1_B]$ in H^B . Hence if $[c_1] = [1_A]$ or $[c_2] = [1_B]$, then $\zeta_A(t) = \zeta_B(t)$. This shows that the two-sided Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are almost conjugate (cf. [5, Theorem 9.3.2]).

Corollary 6.8 (cf. [8, Theorem 3.5]). *Suppose that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent. Then we have $\det(\text{id} - A) = \det(\text{id} - B)$.*

Proof. We know $\zeta_{[c_1]}(t) = \zeta_{\bar{\sigma}_A, \bar{c}_1}(s)$ for $t = e^{-s}$. As $\zeta_B(t) = \zeta_{[c_1]}(t)$, by putting $s = 0$ in (6.9), we get $\det(\text{id} - B) = \det(\text{id} - A)$. \square

The above corollary has been already shown in [8, Theorem 3.5] by using [1] and [12]. Our proof in this paper is direct without using their results.

7 Invariant measures

In this section, we will show that the set of σ_A -invariant measures on X_A is invariant under the continuous orbit equivalence class of one-sided topological Markov shift (X_A, σ_A) . Throughout the section, a measure means a regular Borel measure. We denote by $M(X_A, \sigma_A)$, $M(X_A, \sigma_A)_+$ and $P(X_A, \sigma_A)$ the set of σ_A -invariant measures on X_A , the set of σ_A -invariant positive measures on X_A and the set of σ_A -invariant probability measures on X_A respectively. We identify a regular Borel measure on X_A with a continuous linear functional on the commutative C^* -algebra $C(X_A, \mathbb{C})$ of \mathbb{C} -valued continuous functions on X_A . By the identification, one may write

$$\begin{aligned} M(X_A, \sigma_A) &= \{\varphi \in C(X_A, \mathbb{C})^* \mid \varphi(f \circ \sigma_A) = \varphi(f) \text{ for all } f \in C(X_A, \mathbb{C})\}, \\ M(X_A, \sigma_A)_+ &= \{\varphi \in M(X_A, \sigma_A) \mid \varphi(f) \geq 0 \text{ for all } f \in C(X_A, \mathbb{C}) \text{ with } f \geq 0\}, \\ P(X_A, \sigma_A) &= \{\varphi \in M(X_A, \sigma_A)_+ \mid \varphi(1) = 1\}. \end{aligned}$$

Let h be a homomorphism which gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) . For $f \in C(X_B, \mathbb{C})$ and $x \in X_A$, let us define $\Psi_h(f)(x)$ by the same formula as (4.2). We use the same notation Ψ_h as the previous sections without confusions.

Lemma 7.1. *Keep the above notations. The map $\Psi_h : C(X_B, \mathbb{C}) \longrightarrow C(X_A, \mathbb{C})$ is a continuous linear map such that*

- (i) $\varphi \circ \Psi_h \in M(X_B, \sigma_B)$ for $\varphi \in M(X_A, \sigma_A)$.
- (ii) $\varphi \circ \Psi_h \in M(X_B, \sigma_B)_+$ for $\varphi \in M(X_A, \sigma_A)_+$.
- (iii) If in particular $[c_1] = [1]$ in H^A , we have $\varphi \circ \Psi_h \in P(X_B, \sigma_B)$ for $\varphi \in P(X_A, \sigma_A)$.

Proof. (i) Since $l_1, k_1 \in C(X_A, \mathbb{Z}_+)$, there exists $0 < M \in \mathbb{R}$ such that $\sup_{x \in X_A} (l_1(x) + k_1(x)) \leq M$, so that it is easy to see that the inequality $\|\Psi_h(f)\| \leq M\|f\|$ for $f \in C(X_A, \mathbb{C})$ holds. Hence $\Psi_h : C(X_B, \mathbb{C}) \longrightarrow C(X_A, \mathbb{C})$ is a continuous linear map. We note that the

same equality as in Lemma 4.6 holds for $f \in C(X_B, \mathbb{C})$ by its proof. For $\varphi \in M(X_A, \sigma_A)$ and $f \in C(X_B, \mathbb{C})$, it follows that

$$\begin{aligned}\varphi(\Psi_h(f \circ \sigma_B)) &= \varphi(\Psi_h(f \circ \sigma_B - f) + \Psi_h(f)) \\ &= \varphi(f \circ h \circ \sigma_A - f \circ h) + \varphi(\Psi_h(f)) \\ &= \varphi(\Psi_h(f))\end{aligned}$$

so that $\varphi \circ \Psi_h$ is σ_B -invariant.

(ii) We next assume $\varphi \in M(X_A, \sigma_A)_+$ a positive measure on X_A . As $\bar{\Psi}_h(H_+^B) \subset H_+^A$, for $f \in C(X_B, \mathbb{Z}_+)$ there exist $f_o \in C(X_A, \mathbb{Z}_+)$, $g_o \in C(X_B, \mathbb{Z})$ such that

$$\Psi_h(f) = f_o + g_o - g_o \circ \sigma_A$$

so that

$$\varphi(\Psi_h(f)) = \varphi(f_o + g_o - g_o \circ \sigma_A) = \varphi(f_o) \geq 0.$$

Let us next consider a nonnegative real valued function f on X_B . It is written $f = \sum_{i=1}^n r_i \chi_{U_{\mu(i)}}$ for some $0 \leq r_i \in \mathbb{R}$ and $\mu(i) \in B_*(X_B)$, $i = 1, \dots, n$, where $\chi_{U_{\mu(i)}}$ is the characteristic function of the cylinder set $U_{\mu(i)}$ for the word $\mu(i)$. Since Ψ_h is linear and $\chi_{U_{\mu(i)}} \in C(X_B, \mathbb{Z}_+)$, one has $\Psi_h(\chi_{U_{\mu(i)}}) \geq 0$ as above so that

$$\varphi(\Psi_h(f)) = \sum_{i=1}^n r_i \varphi(\chi_{U_{\mu(i)}}) \geq 0$$

and hence we have $\varphi \circ \Psi_h \in M(X_A, \sigma_A)_+$.

(iii) If $[c_1] = [1]$ in H^A , we have

$$\varphi(\Psi_h(1)) = \varphi(c_1) = \varphi(1).$$

This implies that $\varphi \circ \Psi_h \in P(X_A, \sigma_A)_+$ for $\varphi \in P(X_B, \sigma_B)_+$. \square

Theorem 7.2. *Suppose that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent through a homeomorphism $h : X_A \rightarrow X_B$. Then the previously defined isomorphisms*

$$\Psi_h : C(X_B, \mathbb{Z}) \rightarrow C(X_A, \mathbb{Z}), \quad \Psi_{h^{-1}} : C(X_A, \mathbb{Z}) \rightarrow C(X_B, \mathbb{Z})$$

extend to continuous linear maps between the Banach spaces

$$\Psi_h : C(X_B, \mathbb{C}) \rightarrow C(X_A, \mathbb{C}), \quad \Psi_{h^{-1}} : C(X_A, \mathbb{C}) \rightarrow C(X_B, \mathbb{C})$$

which are inverses to each other such that

- (1) Ψ_h (resp. $\Psi_{h^{-1}}$) maps the σ_A (resp. σ_B)-invariant regular Borel positive measures on X_A (resp. X_B) to the σ_B (resp. σ_A)-invariant regular Borel positive measures on X_B (resp. X_A).
- (2) If in particular, the class $[c_1]$ (resp. $[c_2]$) of the cocycle function c_1 (resp. c_2) is cohomologous to 1 in H^A (resp. H^B), Ψ_h (resp. $\Psi_{h^{-1}}$) maps the σ_A (resp. σ_B)-invariant regular Borel probability measures on X_A (resp. X_B) to the σ_B (resp. σ_A)-invariant regular Borel probability measures on X_B (resp. X_A).

Hence the set of the shift-invariant regular Borel measures on the one-sided shift space is invariant under continuous orbit equivalence of one-sided topological Markov shifts.

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